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Fuzzy risk analysis based on measures of similarity between interval-valued fuzzy numbers

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Abstract

In this paper, we present a new method for handling fuzzy risk analysis problems based on measures of similarity between interval-valued fuzzy numbers. First, we propose a similarity measure to calculate the degree of similarity between interval-valued fuzzy numbers. The proposed similarity measure uses the concept of geometry to calculate the center-of-gravity (COG) points of the lower fuzzy numbers and the upper fuzzy numbers of interval-valued fuzzy numbers, respectively, to calculate the degree of similarity between interval-valued fuzzy numbers. We also prove some properties of the proposed similarity measure. Then, we use the proposed similarity measure for interval-valued fuzzy numbers for handling fuzzy risk analysis problems. The proposed method is more flexible and more intelligent than the methods presented in [S.J. Chen, S.M. Chen, Fuzzy risk analysis based on similarity measures of generalized fuzzy numbers, IEEE Transactions on Fuzzy Systems 11 (1) (2003) 45–56; S.M. Chen, Evaluating the rate of aggregative risk in software development using fuzzy set theory, Cybernetics and Systems 30 (1) (1999) 57–75; S.M. Chen, New methods for subjective mental workload assessment and fuzzy risk analysis, Cybernetics and Systems 27 (5) (1996) 449–472; H.M. Lee, Applying fuzzy set theory to evaluate the rate of aggregative risk in software development, Fuzzy Sets and Systems 79 (3) (1996) 323–336; K.J. Schmucker, Fuzzy Sets, Natural Language Computations, and Risk Analysis, Computer Science Press, MD (1984)] due to the fact that it uses interval-valued fuzzy numbers rather than fuzzy numbers or generalized fuzzy numbers for handling fuzzy risk analysis problems. It provides us with a useful way for handling fuzzy risk analysis problems.

Keywords: Similarity measures; Interval-valued fuzzy numbers; Center-of-gravity points; Fuzzy risk analysis

1. Introduction

Interval-valued fuzzy numbers are very useful to represent evaluating values in real-world problems. In [6], Guijun and Xiaoping defined the interval-valued fuzzy numbers and their extended operations. In [7], Wang and Li presented the concept of correlation coefficients of interval-valued fuzzy numbers and studied some of their properties. In [8], Lin used interval-valued fuzzy numbers for representing vague processing times for handling job-shop scheduling problems. In [9], Hong and Lee presented a distance measure for interval-valued fuzzy numbers. In [10], we presented a method for handling information filtering problems based on interval-valued fuzzy numbers.

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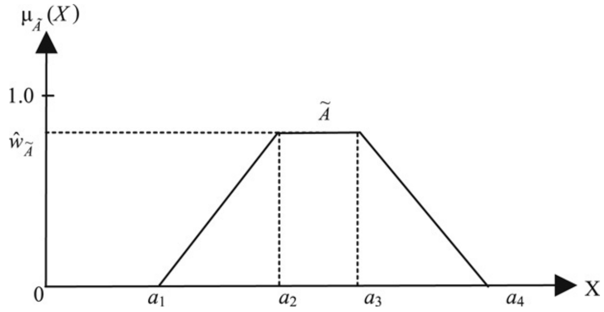

Fig. 1. A generalized trapezoidal fuzzy number $\tilde{A}$.

In [1], we presented a method for fuzzy risk analysis based on similarity measures of generalized fuzzy numbers [11]. In [2], Chen presented a method for evaluating the rate of aggregative risk in software development using the fuzzy set theory. In [3], Chen presented the methods for subjective workload assessment and fuzzy risk analysis. In [4], Lee presented a method for applying the fuzzy set theory to evaluate the rate of aggregative risk in software development. In [5], Schmucker presented a fuzzy risk analyzer (FRA) based on the arithmetic operations of fuzzy numbers.

In recent years, some methods have been presented to calculate the degree of similarity between fuzzy numbers [1,3,12–14]. However, these similarity measures cannot calculate the degree of similarity between interval-valued fuzzy numbers. In this paper, we present a new similarity measure between interval-valued fuzzy numbers and present a new method for handling fuzzy risk analysis problems based on the proposed similarity measure between interval-valued fuzzy numbers. First, we propose a similarity measure to calculate the degree of similarity between interval-valued fuzzy numbers. The proposed similarity measure uses the concept of geometry to calculate the center-of-gravity (COG) points of the lower fuzzy numbers and the upper fuzzy numbers of interval-valued fuzzy numbers, respectively, to calculate the degree of similarity between interval-valued fuzzy numbers. We also prove some properties of the proposed similarity measure. Then, we use the proposed similarity measure of interval-valued fuzzy numbers for handling fuzzy risk analysis problems. The proposed method is more flexible and more intelligent than the methods presented in [1–5] for handling fuzzy risk analysis problems due to the fact that it uses interval-valued fuzzy numbers rather than fuzzy numbers or generalized fuzzy numbers to represent the probability of failure of each sub-component and the severity of loss of each sub-component, respectively.

The rest of this paper is organized as follows. In Section 2, we briefly review the definitions of generalized trapezoidal fuzzy numbers [11,15], interval-valued fuzzy sets [16,17], interval-valued fuzzy numbers [9] and the simple center-of-gravity method (SCGM) [1]. In Section 3, we present a new similarity measure between interval-valued fuzzy numbers. Furthermore, we also prove some properties of the proposed similarity measure and use some examples to illustrate the process of calculating the degree of similarity between interval-valued fuzzy numbers. In Section 4, we use the proposed method for handling fuzzy risk analysis problems. The conclusions are discussed in Section 5.

2. Preliminaries

In this section, we briefly review the definitions of generalized trapezoidal fuzzy numbers [11,15], interval-valued fuzzy sets [16,17], interval-valued fuzzy numbers [9] and the simple center-of-gravity method (SCGM) [1].

In [11,15], Chen represented a generalized trapezoidal fuzzy number $\tilde{A}$ as $\tilde{A} = (a_1, a_2, a_3, a_4; \hat{w}_{\tilde{A}})$, where $0 < \hat{w}_{\tilde{A}} \leq 1$ and a_1, a_2, a_3 and a_4 are real numbers. A generalized fuzzy number $\tilde{A}$ of the universe of discourse X is characterized by a membership function $\mu_{\tilde{A}}$, where $\mu_{\tilde{A}}: X \to [0, 1]$, as shown in Fig. 1.

Definition 1. An interval-valued fuzzy set C defined in the universe of discourse X is given by

$$C = \{(x, [\mu_C^L(x), \mu_C^U(x)]) | x \in X\},$$

where $0 \leq \mu_C^L(x) \leq \mu_C^U(x) \leq 1$ and the membership grade $\mu_C(x)$ of the element x belongs to the interval-valued fuzzy set C, which can be represented by an interval $\mu_C(x) = [\mu_C^L(x), \mu_C^U(x)]$.

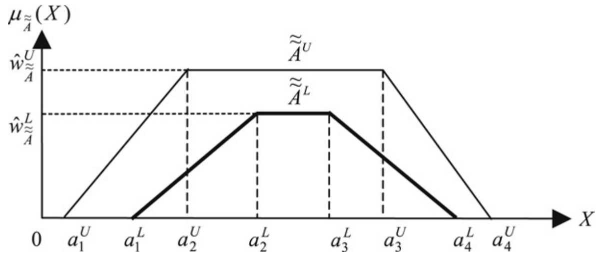

Fig. 2. Interval-valued fuzzy number $\tilde{\tilde{A}}$.

Definition 2 (*[9]*). If an interval-valued fuzzy set $\tilde{\tilde{A}}$ satisfies the following properties:

(1) $\tilde{\tilde{A}}$ is defined in a closed bounded interval,
(2) $\tilde{\tilde{A}}$ is a convex set,

then $\tilde{\tilde{A}}$ is called an interval-valued fuzzy number in the universe of discourse X.

Assume that there is an interval-valued fuzzy number $\tilde{\tilde{A}} = [\tilde{\tilde{A}}^L, \tilde{\tilde{A}}^U]$ as shown in Fig. 2. From Fig. 2, we can see that the interval-valued fuzzy number $\tilde{\tilde{A}}$ has two elements, where the one is the lower fuzzy number $\tilde{\tilde{A}}^L$, and the other one is the upper fuzzy number $\tilde{\tilde{A}}^U$. Furthermore, the interval-valued fuzzy number $\tilde{\tilde{A}}$ can be represented as $\tilde{\tilde{A}} = [(a_1^L, a_2^L, a_3^L, a_4^L; \hat{w}^L_{\tilde{\tilde{A}}}), (a_1^U, a_2^U, a_3^U, a_4^U; \hat{w}^U_{\tilde{\tilde{A}}})]$ where $a_1^U \le a_2^U \le a_3^U \le a_4^U$, $a_1^L \le a_2^L \le a_3^L \le a_4^L$, $0 < \hat{w}^L_{\tilde{\tilde{A}}} \le \hat{w}^U_{\tilde{\tilde{A}}} \le 1$ and $\tilde{\tilde{A}}^L \subset \tilde{\tilde{A}}^U$. If $a_1^L = a_1^U$, $a_2^L = a_2^U$, $a_3^L = a_3^U$, $a_4^L = a_4^U$ and $\hat{w}^L_{\tilde{\tilde{A}}} = \hat{w}^U_{\tilde{\tilde{A}}} = \hat{w}_{\tilde{A}}$, then the interval-valued fuzzy number $\tilde{\tilde{A}}$ can be regarded as a generalized fuzzy number, denoted as $\tilde{A} = (a_1, a_2, a_3, a_4; \hat{w}_{\tilde{A}})$.

Assume that there are two interval-valued fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$, where $\tilde{\tilde{A}} = [(a_1^L, a_2^L, a_3^L, a_4^L; \hat{w}^L_{\tilde{\tilde{A}}}), (a_1^U, a_2^U, a_3^U, a_4^U; \hat{w}^U_{\tilde{\tilde{A}}})]$ and $\tilde{\tilde{B}} = [(b_1^L, b_2^L, b_3^L, b_4^L; \hat{w}^L_{\tilde{\tilde{B}}}), (b_1^U, b_2^U, b_3^U, b_4^U; \hat{w}^U_{\tilde{\tilde{B}}})]$. The arithmetic operations between the interval-valued fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ are as follows [9]:

(1) Interval-Valued Fuzzy Number Addition $\oplus$:

$$\begin{aligned}\tilde{\tilde{A}} \oplus \tilde{\tilde{B}} &= [(a_1^L, a_2^L, a_3^L, a_4^L; \hat{w}^L_{\tilde{\tilde{A}}}), (a_1^U, a_2^U, a_3^U, a_4^U; \hat{w}^U_{\tilde{\tilde{A}}})] \oplus [(b_1^L, b_2^L, b_3^L, b_4^L; \hat{w}^L_{\tilde{\tilde{B}}}), (b_1^U, b_2^U, b_3^U, b_4^U; \hat{w}^U_{\tilde{\tilde{B}}})]\\ &= [(a_1^L + b_1^L, a_2^L + b_2^L, a_3^L + b_3^L, a_4^L + b_4^L; \mathrm{Min}(\hat{w}^L_{\tilde{\tilde{A}}}, \hat{w}^L_{\tilde{\tilde{B}}})),\\ &\quad (a_1^U + b_1^U, a_2^U + b_2^U, a_3^U + b_3^U, a_4^U + b_4^U; \mathrm{Min}(\hat{w}^U_{\tilde{\tilde{A}}}, \hat{w}^U_{\tilde{\tilde{B}}}))], \end{aligned} \tag{1}$$

where $a_1^L, a_2^L, a_3^L, a_4^L, a_1^U, a_2^U, a_3^U, a_4^U, b_1^L, b_2^L, b_3^L, b_4^L, b_1^U, b_2^U, b_3^U$ and b_4^U are any real values, $0 < \hat{w}^L_{\tilde{\tilde{A}}} \le \hat{w}^U_{\tilde{\tilde{A}}} \le 1$, and $0 < \hat{w}^L_{\tilde{\tilde{B}}} \le \hat{w}^U_{\tilde{\tilde{B}}} \le 1$.

(2) Interval-Valued Fuzzy Number Subtraction $\ominus$:

$$\begin{aligned}\tilde{\tilde{A}} \ominus \tilde{\tilde{B}} &= [(a_1^L, a_2^L, a_3^L, a_4^L; \hat{w}^L_{\tilde{\tilde{A}}}), (a_1^U, a_2^U, a_3^U, a_4^U; \hat{w}^U_{\tilde{\tilde{A}}})] \ominus [(b_1^L, b_2^L, b_3^L, b_4^L; \hat{w}^L_{\tilde{\tilde{B}}}), (b_1^U, b_2^U, b_3^U, b_4^U; \hat{w}^U_{\tilde{\tilde{B}}})]\\ &= [(a_1^L - b_4^L, a_2^L - b_3^L, a_3^L - b_2^L, a_4^L - b_1^L; \mathrm{Min}(\hat{w}^L_{\tilde{\tilde{A}}}, \hat{w}^L_{\tilde{\tilde{B}}})),\\ &\quad (a_1^U - b_4^U, a_2^U - b_3^U, a_3^U - b_2^U, a_4^U - b_1^U; \mathrm{Min}(\hat{w}^U_{\tilde{\tilde{A}}}, \hat{w}^U_{\tilde{\tilde{B}}}))], \end{aligned} \tag{2}$$

where $a_1^L, a_2^L, a_3^L, a_4^L, a_1^U, a_2^U, a_3^U, a_4^U, b_1^L, b_2^L, b_3^L, b_4^L, b_1^U, b_2^U, b_3^U$ and b_4^U are any real values, $0 < \hat{w}^L_{\tilde{\tilde{A}}} \le \hat{w}^U_{\tilde{\tilde{A}}} \le 1$, and $0 < \hat{w}^L_{\tilde{\tilde{B}}} \le \hat{w}^U_{\tilde{\tilde{B}}} \le 1$.

(3) Interval-Valued Fuzzy Number Multiplication $\otimes$:

$$\begin{aligned}\tilde{\tilde{A}} \otimes \tilde{\tilde{B}} &= [(a_1^L, a_2^L, a_3^L, a_4^L; \hat{w}_{\tilde{\tilde{A}}}^L), (a_1^U, a_2^U, a_3^U, a_4^U; \hat{w}_{\tilde{\tilde{A}}}^U)] \otimes [(b_1^L, b_2^L, b_3^L, b_4^L; \hat{w}_{\tilde{\tilde{B}}}^L), (b_1^U, b_2^U, b_3^U, b_4^U; \hat{w}_{\tilde{\tilde{B}}}^U)] \\ &= [(a_1^L \times b_1^L, a_2^L \times b_2^L, a_3^L \times b_3^L, a_4^L \times b_4^L; \mathrm{Min}(\hat{w}_{\tilde{\tilde{A}}}^L, \hat{w}_{\tilde{\tilde{B}}}^L)), \\ &\quad (a_1^U \times b_1^U, a_2^U \times b_2^U, a_3^U \times b_3^U, a_4^U \times b_4^U; \mathrm{Min}(\hat{w}_{\tilde{\tilde{A}}}^U, \hat{w}_{\tilde{\tilde{B}}}^U))], \end{aligned} \tag{3}$$

where $a_1^L, a_2^L, a_3^L, a_4^L, a_1^U, a_2^U, a_3^U, a_4^U, b_1^L, b_2^L, b_3^L, b_4^L, b_1^U, b_2^U, b_3^U$ and b_4^U are any real values, $0 < \hat{w}_{\tilde{\tilde{A}}}^L \le \hat{w}_{\tilde{\tilde{A}}}^U \le 1$, and $0 < \hat{w}_{\tilde{\tilde{B}}}^L \le \hat{w}_{\tilde{\tilde{B}}}^U \le 1$.

(4) Interval-Valued Fuzzy Number Division $\oslash$:

$$\begin{aligned}\tilde{\tilde{A}} \oslash \tilde{\tilde{B}} &= [(a_1^L, a_2^L, a_3^L, a_4^L; \hat{w}_{\tilde{\tilde{A}}}^L), (a_1^U, a_2^U, a_3^U, a_4^U; \hat{w}_{\tilde{\tilde{A}}}^U)] \oslash [(b_1^L, b_2^L, b_3^L, b_4^L; \hat{w}_{\tilde{\tilde{B}}}^L), (b_1^U, b_2^U, b_3^U, b_4^U; \hat{w}_{\tilde{\tilde{B}}}^U)] \\ &= [(a_1^L/b_4^L, a_2^L/b_3^L, a_3^L/b_2^L, a_4^L/b_1^L; \mathrm{Min}(\hat{w}_{\tilde{\tilde{A}}}^L, \hat{w}_{\tilde{\tilde{B}}}^L)), \\ &\quad (a_1^U/b_4^U, a_2^U/b_3^U, a_3^U/b_2^U, a_4^U/b_1^U; \mathrm{Min}(\hat{w}_{\tilde{\tilde{A}}}^U, \hat{w}_{\tilde{\tilde{B}}}^U))], \end{aligned} \tag{4}$$

where $a_1^L, a_2^L, a_3^L, a_4^L, a_1^U, a_2^U, a_3^U, a_4^U, b_1^L, b_2^L, b_3^L, b_4^L, b_1^U, b_2^U, b_3^U$ and b_4^U are all nonzero positive real numbers or all nonzero negative real numbers, $0 < \hat{w}_{\tilde{\tilde{A}}}^L \le \hat{w}_{\tilde{\tilde{A}}}^U \le 1$ and $0 < \hat{w}_{\tilde{\tilde{B}}}^L \le \hat{w}_{\tilde{\tilde{B}}}^U \le 1$.

In [1], Chen and Chen presented a simple center-of-gravity method (SCGM) to calculate the center-of-gravity point (x^*, y^*) of a generalized fuzzy number, based on the concept of geometry. Let $\tilde{A}$ be a generalized fuzzy number, $\tilde{A} = (a_1, a_2, a_3, a_4; w_{\tilde{A}})$. The formula for calculating the COG point (x^*, y^*) of the generalized fuzzy number $\tilde{A}$ is as follows:

$$y_{\tilde{A}}^* = \begin{cases} \dfrac{w_{\tilde{A}} \times \left(\frac{a_3 - a_2}{a_4 - a_1} + 2\right)}{6}, & \text{if } a_1 \ne a_4 \text{ and } 0 < w_{\tilde{A}} \le 1, \\ \dfrac{w_{\tilde{A}}}{2}, & \text{if } a_1 = a_4 \text{ and } 0 < w_{\tilde{A}} \le 1, \end{cases} \tag{5}$$

$$x_{\tilde{A}}^* = \frac{y_{\tilde{A}}^*(a_3 + a_2) + (a_4 + a_1)(w_{\tilde{A}} - y_{\tilde{A}}^*)}{2w_{\tilde{A}}}. \tag{6}$$

In [1], we presented a method to evaluate the degree of similarity between generalized fuzzy numbers. Assume that there are two generalized fuzzy numbers $\tilde{A}$ and $\tilde{B}$, where $\tilde{A} = (a_1, a_2, a_3, a_4; \hat{w}_{\tilde{A}})$, $\tilde{B} = (b_1, b_2, b_3, b_4; \hat{w}_{\tilde{B}})$, $0 \le a_1 \le a_2 \le a_3 \le a_4 \le 1$, and $0 \le b_1 \le b_2 \le b_3 \le b_4 \le 1$. First, we use formulas (5) and (6) to obtain the COG points $\mathrm{COG}(\tilde{A})$ and $\mathrm{COG}(\tilde{B})$ of the generalized fuzzy numbers $\tilde{A}$ and $\tilde{B}$, respectively, where $\mathrm{COG}(\tilde{A}) = (x_{\tilde{A}}^*, y_{\tilde{A}}^*)$ and $\mathrm{COG}(\tilde{B}) = (x_{\tilde{B}}^*, y_{\tilde{B}}^*)$. Then, the degree of similarity $S(\tilde{A}, \tilde{B})$ between the generalized fuzzy numbers $\tilde{A}$ and $\tilde{B}$ can be calculated as follows:

$$S(\tilde{A}, \tilde{B}) = \left[\left(1 - \frac{\sum_{i=1}^{4} |a_i - b_i|}{4}\right) \times (1 - |x_{\tilde{A}}^* - x_{\tilde{B}}^*|)\right]^{\frac{1}{2}} \times \frac{\min(y_{\tilde{A}}^*, y_{\tilde{B}}^*)}{\max(y_{\tilde{A}}^*, y_{\tilde{B}}^*)}, \tag{7}$$

where the values of $y_{\tilde{A}}^*$ and $y_{\tilde{B}}^*$ are obtained by formula (5), the values of $x_{\tilde{A}}^*$ and $x_{\tilde{B}}^*$ by formula (6), and $S(\tilde{A}, \tilde{B}) \in [0, 1]$. The larger the value of $S(\tilde{A}, \tilde{B})$, the higher the similarity between the generalized fuzzy numbers $\tilde{A}$ and $\tilde{B}$.

3. A similarity measure between interval-valued fuzzy numbers

In this section, we propose a similarity measure to calculate the degree of similarity between interval-valued fuzzy numbers. Let U be the universe of discourse, $U = [0, 1]$. Assume that there are two interval-valued fuzzy numbers $\tilde{\tilde{A}} = [\tilde{\tilde{A}}^L, \tilde{\tilde{A}}^U] = [(a_1^L, a_2^L, a_3^L, a_4^L; \hat{w}_{\tilde{\tilde{A}}}^L), (a_1^U, a_2^U, a_3^U, a_4^U; \hat{w}_{\tilde{\tilde{A}}}^U)]$ and $\tilde{\tilde{B}} = [\tilde{\tilde{B}}^L, \tilde{\tilde{B}}^U] = [(b_1^L, b_2^L, b_3^L, b_4^L; \hat{w}_{\tilde{\tilde{B}}}^L), (b_1^U, b_2^U, b_3^U, b_4^U; \hat{w}_{\tilde{\tilde{B}}}^U)]$, where $0 \le a_1^L \le a_2^L \le a_3^L \le a_4^L \le 1$, $0 \le a_1^U \le a_2^U \le a_3^U \le a_4^U \le 1$, $0 < \hat{w}_{\tilde{\tilde{A}}}^L \le \hat{w}_{\tilde{\tilde{A}}}^U \le 1$ and $\tilde{\tilde{A}}^L \subset \tilde{\tilde{A}}^U$; $0 \le b_1^L \le b_2^L \le b_3^L \le b_4^L \le 1$, $0 \le b_1^U \le b_2^U \le b_3^U \le b_4^U \le 1$, $0 < \hat{w}_{\tilde{\tilde{B}}}^L \le \hat{w}_{\tilde{\tilde{B}}}^U \le 1$ and $\tilde{\tilde{B}}^L \subset \tilde{\tilde{B}}^U$. The proposed method is now presented as follows:

Step 1 Based on formulas (5) and (6), get the COG points of the interval-valued fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$, respectively. For example, the interval-valued fuzzy number $\tilde{\tilde{A}}$ has two different COG points, where the one is the COG point $(x^*_{\tilde{\tilde{A}}^L}, y^*_{\tilde{\tilde{A}}^L})$ of the lower fuzzy number $\tilde{\tilde{A}}^L$, and the other one is the COG point $(x^*_{\tilde{\tilde{A}}^U}, y^*_{\tilde{\tilde{A}}^U})$ of the upper fuzzy number $\tilde{\tilde{A}}^U$, shown as follows:

$$y^*_{\tilde{\tilde{A}}^L} = \begin{cases} \dfrac{\hat{w}_{\tilde{\tilde{A}}}^L \times \left(\frac{a_3^L - a_2^L}{a_4^L - a_1^L} + 2\right)}{6}, & \text{if } a_1^L \ne a_4^L \text{ and } 0 < \hat{w}_{\tilde{\tilde{A}}}^L \le 1, \\ \dfrac{\hat{w}_{\tilde{\tilde{A}}}^L}{2}, & \text{if } a_1^L = a_4^L \text{ and } 0 < \hat{w}_{\tilde{\tilde{A}}}^L \le 1, \end{cases} \tag{8}$$

$$x^*_{\tilde{\tilde{A}}^L} = \frac{y^*_{\tilde{\tilde{A}}^L}(a_3^L + a_2^L) + (a_4^L + a_1^L)(\hat{w}_{\tilde{\tilde{A}}}^L - y^*_{\tilde{\tilde{A}}^L})}{2\hat{w}_{\tilde{\tilde{A}}}^L}, \tag{9}$$

$$y^*_{\tilde{\tilde{A}}^U} = \begin{cases} \dfrac{\hat{w}_{\tilde{\tilde{A}}}^U \times \left(\frac{a_3^U - a_2^U}{a_4^U - a_1^U} + 2\right)}{6}, & \text{if } a_1^U \ne a_4^U \text{ and } 0 < \hat{w}_{\tilde{\tilde{A}}}^U \le 1, \\ \dfrac{\hat{w}_{\tilde{\tilde{A}}}^U}{2}, & \text{if } a_1^U = a_4^U \text{ and } 0 < \hat{w}_{\tilde{\tilde{A}}}^U \le 1, \end{cases} \tag{10}$$

$$x^*_{\tilde{\tilde{A}}^U} = \frac{y^*_{\tilde{\tilde{A}}^U}(a_3^U + a_2^U) + (a_4^U + a_1^U)(\hat{w}_{\tilde{\tilde{A}}}^U - y^*_{\tilde{\tilde{A}}^U})}{2\hat{w}_{\tilde{\tilde{A}}}^U}. \tag{11}$$

In the same way, we can get the COG points $(x^*_{\tilde{\tilde{B}}^L}, y^*_{\tilde{\tilde{B}}^L})$ of the lower fuzzy number $\tilde{\tilde{B}}^L$ and $(x^*_{\tilde{\tilde{B}}^U}, y^*_{\tilde{\tilde{B}}^U})$ of the upper fuzzy number $\tilde{\tilde{B}}^U$ of the interval-valued fuzzy number $\tilde{\tilde{B}}$, respectively.

Step 2 Based on formula (7), calculate the degree of similarity $S(\tilde{\tilde{A}}^L, \tilde{\tilde{B}}^L)$ between the lower fuzzy numbers $\tilde{\tilde{A}}^L$ and $\tilde{\tilde{B}}^L$ and calculate the degree of similarity $S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U)$ between the upper fuzzy numbers $\tilde{\tilde{A}}^U$ and $\tilde{\tilde{B}}^U$, respectively, as follows:

$$S(\tilde{\tilde{A}}^L, \tilde{\tilde{B}}^L) = \left[\left(1 - \frac{\sum_{i=1}^{4} \left|a_i^L - b_i^L\right|}{4}\right) \times \left(1 - \left|x^*_{\tilde{\tilde{A}}^L} - x^*_{\tilde{\tilde{B}}^L}\right|\right)\right]^{\frac{1}{2}} \times \frac{\min(y^*_{\tilde{\tilde{A}}^L}, y^*_{\tilde{\tilde{B}}^L})}{\max(y^*_{\tilde{\tilde{A}}^L}, y^*_{\tilde{\tilde{B}}^L})}, \tag{12}$$

$$S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U) = \left[\left(1 - \frac{\sum_{i=1}^{4} \left|a_i^U - b_i^U\right|}{4}\right) \times (1 - |x^*_{\tilde{\tilde{A}}^U} - x^*_{\tilde{\tilde{B}}^U}|)\right]^{\frac{1}{2}} \times \frac{\min(y^*_{\tilde{\tilde{A}}^U}, y^*_{\tilde{\tilde{B}}^U})}{\max(y^*_{\tilde{\tilde{A}}^U}, y^*_{\tilde{\tilde{B}}^U})}, \tag{13}$$

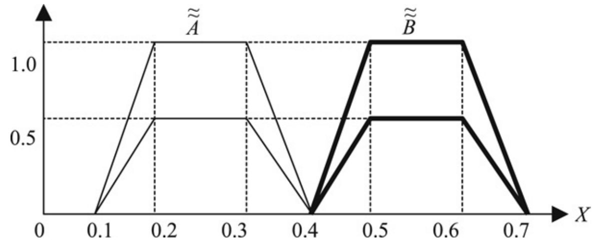

Fig. 3. Interval-valued fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ of Example 1.

where $S(\tilde{A}^L, \tilde{B}^L) \in [0, 1]$ and $S(\tilde{A}^U, \tilde{B}^U) \in [0, 1]$. The larger the value of $S(\tilde{A}^L, \tilde{B}^L)$, the higher the similarity between the lower fuzzy numbers $\tilde{A}^L$ and $\tilde{B}^L$; the larger the value of $S(\tilde{A}^U, \tilde{B}^U)$ the higher the similarity between the upper fuzzy numbers $\tilde{A}^U$ and $\tilde{B}^U$.

Step 3 Calculate the degree of similarity $S(\tilde{\tilde{A}}, \tilde{\tilde{B}})$ between the interval-valued fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ as follows:

$$S(\tilde{\tilde{A}}, \tilde{\tilde{B}}) = \sqrt{S(\tilde{A}^L, \tilde{B}^L) \times S(\tilde{A}^U, \tilde{B}^U)}, \tag{14}$$

where $S(\tilde{\tilde{A}}, \tilde{\tilde{B}}) \in [0, 1]$. The larger the value of $S(\tilde{\tilde{A}}, \tilde{\tilde{B}})$, the higher the similarity between the interval-valued fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$.

The proposed similarity measure between interval-valued fuzzy numbers has the following properties, where the proofs of these properties are shown in Appendix:

Property 1. *Two interval-valued fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ are identical if and only if* $S(\tilde{\tilde{A}}, \tilde{\tilde{B}}) = 1$.

Property 2. $S(\tilde{\tilde{A}}, \tilde{\tilde{B}}) = S(\tilde{\tilde{B}}, \tilde{\tilde{A}})$.

Property 3. *If $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ are two real numbers, then* $S(\tilde{\tilde{A}}, \tilde{\tilde{B}}) = 1 - |a - b|$.

In the following, we use two examples to illustrate the process of calculating the degrees of similarity between interval-valued fuzzy numbers.

Example 1. Assume that there are two interval-valued fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$, where

$$\tilde{\tilde{A}} = [(0.1, 0.2, 0.3, 0.4; 0.5), (0.1, 0.2, 0.3, 0.4; 1)],$$
$$\tilde{\tilde{B}} = [(0.4, 0.5, 0.6, 0.7; 0.5), (0.4, 0.5, 0.6, 0.7; 1)],$$

as shown in Fig. 3.

The process for calculating the degree of similarity between the interval-valued fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ is as follows:

Step 1 Based on formulas (8)–(11), we can get the COG point $\mathrm{COG}(\tilde{A}^L) = (x^*_{\tilde{A}^L}, y^*_{\tilde{A}^L})$ of the lower fuzzy number $\tilde{A}^L$, shown as follows:

$$\begin{aligned} y^*_{\tilde{A}^L} &= \frac{\hat{w}^L_{\tilde{A}} \times \left(\frac{a^L_3 - a^L_2}{a^L_4 - a^L_1} + 2\right)}{6} \\ &= \frac{0.5 \times \left(\frac{0.3-0.2}{0.4-0.1} + 2\right)}{6} \\ &= 0.1944, \end{aligned}$$

$$
\begin{aligned}
x^*_{\tilde{\tilde{A}}^L} &= \frac{y^*_{\tilde{\tilde{A}}^L}(a_3^L + a_2^L) + (a_4^L + a_1^L)(\hat{w}^L_{\tilde{\tilde{A}}} - y^*_{\tilde{\tilde{A}}^L})}{2\hat{w}^L_{\tilde{\tilde{A}}}} \\
&= \frac{0.1944 \times (0.3 + 0.2) + (0.4 + 0.1) \times (0.5 - 0.1944)}{2 \times 0.5} \\
&= 0.25.
\end{aligned}
$$

In the same way, we can get the COG point $\mathrm{COG}(\tilde{\tilde{B}}^L)$ of the lower fuzzy number $\tilde{\tilde{B}}^L$, the COG point $\mathrm{COG}(\tilde{\tilde{A}}^U)$ of the upper fuzzy number $\tilde{\tilde{A}}^U$, and the COG point $\mathrm{COG}(\tilde{\tilde{B}}^U)$ of the upper fuzzy number $\tilde{\tilde{B}}^U$, where $\mathrm{COG}(\tilde{\tilde{B}}^L) = (0.55, 0.1944)$, $\mathrm{COG}(\tilde{\tilde{A}}^U) = (0.25, 0.3889)$, and $\mathrm{COG}(\tilde{\tilde{B}}^U) = (0.55, 0.3889)$.

Step 2 Based on formula (12), we can calculate the degree of similarity $S(\tilde{\tilde{A}}^L, \tilde{\tilde{B}}^L)$ between the lower fuzzy numbers $\tilde{\tilde{A}}^L$ and $\tilde{\tilde{B}}^L$, shown as follows:

$$
\begin{aligned}
S(\tilde{\tilde{A}}^L, \tilde{\tilde{B}}^L) &= \left[\left(1 - \frac{\sum_{i=1}^{4} \left|a_i^L - b_i^L\right|}{4}\right) \times \left(1 - \left|x^*_{\tilde{\tilde{A}}^L} - x^*_{\tilde{\tilde{B}}^L}\right|\right)\right]^{\frac{1}{2}} \times \frac{\min(y^*_{\tilde{\tilde{A}}^L}, y^*_{\tilde{\tilde{B}}^L})}{\max(y^*_{\tilde{\tilde{A}}^L}, y^*_{\tilde{\tilde{B}}^L})} \\
&= \left[\left(1 - \frac{|0.1 - 0.4| + |0.2 - 0.5| + |0.3 - 0.6| + |0.4 - 0.7|}{4}\right)\right. \\
&\quad \left. \times (1 - |0.25 - 0.55|) \times 1\right] \\
&= 0.7.
\end{aligned}
$$

In the same way, based on formula (13), we can calculate the degree of similarity $S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U)$ between the upper fuzzy numbers $\tilde{\tilde{A}}^U$ and $\tilde{\tilde{B}}^U$, where $S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U) = 0.7$.

Step 3 Based on formula (14), the degree of similarity $S(\tilde{\tilde{A}}, \tilde{\tilde{B}})$ between the interval-valued fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ can be calculated as follows:

$$
\begin{aligned}
S(\tilde{\tilde{A}}, \tilde{\tilde{B}}) &= \sqrt{S(\tilde{\tilde{A}}^L, \tilde{\tilde{B}}^L) \times S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U)} \\
&= \sqrt{0.7 \times 0.7} \\
&= 0.7.
\end{aligned}
$$

That is, the degree of similarity between the interval-valued fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ is about 0.7.

Example 2. Assume that $\tilde{\tilde{A}}$ is an interval-valued fuzzy number and $\tilde{B}$ is a generalized fuzzy number, where

$$
\begin{aligned}
\tilde{\tilde{A}} &= [(0.1, 0.2, 0.3, 0.4; 0.5), (0.1, 0.2, 0.3, 0.4; 1)], \\
\tilde{B} &= (0.4, 0.5, 0.6, 0.7; 0.5),
\end{aligned}
$$

as shown in Fig. 4. Based on the discussions of Section 2, we can see that the generalized fuzzy number $\tilde{B}$ can be represented as follows:

$$
\begin{aligned}
\tilde{B} &= (0.4, 0.5, 0.6, 0.7; 0.5) \\
&= [(0.4; 0.5, 0.6, 0.7; 0.5), (0.4, 0.5, 0.6, 0.7; 0.5)].
\end{aligned}
$$

The process for calculating the degree of similarity between the interval-valued fuzzy number $\tilde{\tilde{A}}$ and the generalized fuzzy number $\tilde{B}$ is as follows:

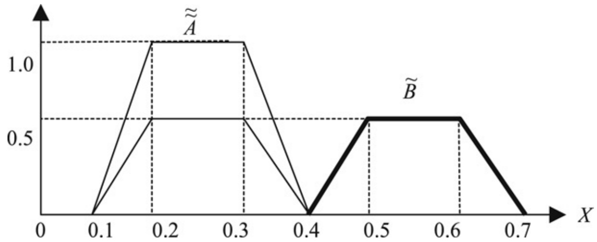

Fig. 4. Interval-valued fuzzy number $\tilde{\tilde{A}}$ and generalized fuzzy number $\tilde{B}$ of Example 2.

[*Step* 1] Based on formulas (8)–(11), we can get the COG point $\text{COG}(\tilde{\tilde{A}}^L) = (x^*_{\tilde{\tilde{A}}^L}, y^*_{\tilde{\tilde{A}}^L})$ of the lower fuzzy number $\tilde{\tilde{A}}^L$, shown as follows:

$$\begin{aligned} y^*_{\tilde{\tilde{A}}^L} &= \frac{\hat{w}^L_{\tilde{\tilde{A}}} \times \left(\frac{a^L_3 - a^L_2}{a^L_4 - a^L_1} + 2\right)}{6} \\ &= \frac{0.5 \times \left(\frac{0.3-0.2}{0.4-0.1} + 2\right)}{6} \\ &= 0.1944, \\ x^*_{\tilde{\tilde{A}}^L} &= \frac{y^*_{\tilde{\tilde{A}}^L}(a^L_3 + a^L_2) + (a^L_4 + a^L_1)(\hat{w}^L_{\tilde{\tilde{A}}} - y^*_{\tilde{\tilde{A}}^L})}{2\hat{w}^L_{\tilde{\tilde{A}}}} \\ &= \frac{0.1944 \times (0.3 + 0.2) + (0.4 + 0.1) \times (0.5 - 0.1944)}{2 \times 0.5} \\ &= 0.25. \end{aligned}$$

In the same way, we can get the COG point $\text{COG}(\tilde{\tilde{A}}^U)$ of the interval-valued fuzzy number $\tilde{\tilde{A}}$, where $\text{COG}(\tilde{\tilde{A}}^U) = (0.25, 0.3889)$. Based on formulas (5) and (6), we can get the COG point $\text{COG}(\tilde{B})$ of the generalized fuzzy number $\tilde{B}$, where $\text{COG}(\tilde{B}) = (0.55, 0.1944)$.

[*Step* 2] Based on formula (12), we can calculate the degree of similarity $S(\tilde{\tilde{A}}^L, \tilde{B})$ between the interval-valued fuzzy number $\tilde{\tilde{A}}$ and the generalized fuzzy number $\tilde{B}$, shown as follows:

$$\begin{aligned} S(\tilde{\tilde{A}}^L, \tilde{B}) &= \left[\left(1 - \frac{\sum_{i=1}^{4} \left|a^L_i - b_i\right|}{4}\right) \times \left(1 - \left|x^*_{\tilde{\tilde{A}}^L} - x^*_{\tilde{B}}\right|\right)\right]^{\frac{1}{2}} \times \frac{\min(y^*_{\tilde{\tilde{A}}^L}, y^*_{\tilde{B}})}{\max(y^*_{\tilde{\tilde{A}}^L}, y^*_{\tilde{B}})} \\ &= \left[\left(1 - \frac{|0.1 - 0.4| + |0.2 - 0.5| + |0.3 - 0.6| + |0.4 - 0.7|}{4}\right)\right. \\ &\qquad \left. \times (1 - |0.25 - 0.55|) \times 1\right] \\ &= 0.7. \end{aligned}$$

In the same way, based on formula (13), we can calculate the degree of similarity $S(\tilde{\tilde{A}}^U, \tilde{B})$ between the interval-valued fuzzy number $\tilde{\tilde{A}}$ and the generalized fuzzy number$\tilde{B}$, where $S(\tilde{\tilde{A}}^U, \tilde{B}) = 0.35$.

[*Step* 3] Based on formula (14), we can calculate the degree of similarity $S(\tilde{\tilde{A}}, \tilde{B})$ between the interval-valued fuzzy number $\tilde{\tilde{A}}$ and the generalized fuzzy number $\tilde{B}$, shown as follows:

$$\begin{aligned} S(\tilde{\tilde{A}}, \tilde{B}) &= \sqrt{S(\tilde{\tilde{A}}^L, \tilde{B}) \times S(\tilde{\tilde{A}}^U, \tilde{B})} \\ &= \sqrt{0.7 \times 0.35} \\ &\risingdotseq 0.495. \end{aligned}$$

Therefore, the degree of similarity between the interval-valued fuzzy number $\tilde{\tilde{A}}$ and the generalized fuzzy number$\tilde{B}$ is about 0.495.

4. Fuzzy risk analysis based on the similarity measure between interval-valued fuzzy numbers

In this section, we apply the proposed similarity measure for interval-valued fuzzy numbers for handling fuzzy risk analysis problems. It is obvious that interval-valued fuzzy numbers are different from generalized fuzzy numbers due to the fact that they have two different elements and these two elements can be regarded as two different generalized fuzzy numbers. In [10], we have pointed out that using interval-valued fuzzy numbers to represent linguistic terms can improve the flexibility of a system. Because the proposed fuzzy risk analysis method uses interval-valued fuzzy numbers to represent the probability of failure of each sub-component and the severity of loss of each sub-component, respectively, it provides us with a useful way for handling fuzzy risk analysis problems in a more flexible and more intelligent manner.

Assume that there is a component $\boldsymbol{A}$ consisting of n sub-components $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_n$, and we want to evaluate the probability of failure of component $\boldsymbol{A}$. Assume that $\tilde{\tilde{R}}_i$ denotes the probability of failure of the sub-component $\boldsymbol{A}_i$, $\tilde{\tilde{W}}_i$ denotes the severity of loss of the sub-component $\boldsymbol{A}_i$, where $\tilde{\tilde{R}}_i$ and $\tilde{\tilde{W}}_i$ are interval-valued fuzzy numbers, and $1 \le i \le n$. The algorithm for fuzzy risk analysis is now presented as follows.

Step 1 Use the fuzzy weighted mean method and the interval-valued fuzzy number arithmetic operations to integrate the evaluating values $\tilde{\tilde{R}}_i$ and $\tilde{\tilde{W}}_i$ of each sub-component $\boldsymbol{A}_i$, where $1 \le i \le n$, to get the total risk $\tilde{\tilde{R}}$ of the component $\boldsymbol{A}$, shown as follows:

$$\tilde{\tilde{R}} = \frac{\sum_{i=1}^{n} \tilde{\tilde{W}}_i \otimes \tilde{\tilde{R}}_i}{\sum_{i=1}^{n} \tilde{\tilde{W}}_i}, \tag{15}$$

where $\tilde{\tilde{R}}$ is an interval-valued fuzzy number.

Step 2 Use the proposed similarity measure to calculate the degree of similarity between the interval-valued fuzzy number $\tilde{\tilde{R}}$ and each linguistic term shown in Table 1. Translate the interval-valued fuzzy number $\tilde{\tilde{R}}$ into a linguistic term shown in Table 1, which has the largest degree of similarity with respect to $\tilde{\tilde{R}}$.

Example 3. Assume that the component $\boldsymbol{A}$ consists of three sub-components $\boldsymbol{A}_1, \boldsymbol{A}_2$ and $\boldsymbol{A}_3$, and we want to evaluate the probability of failure of the component $\boldsymbol{A}$. Assume that there are some evaluating values represented by interval-valued fuzzy numbers as shown in Table 2, where $\tilde{\tilde{W}}_i$ denotes the severity of loss of sub-component $\boldsymbol{A}_i$, $\tilde{\tilde{R}}_i$ denotes the probability of failure of sub-component $\boldsymbol{A}_i$, and $1 \le i \le 3$.

[*Step* 1] Based on formulas (1), (3), (4) and (15), the probability of failure $\tilde{\tilde{R}}$ of the component $\boldsymbol{A}$ can be evaluated as follows:

$$\begin{aligned} \tilde{\tilde{R}} &= [\tilde{\tilde{W}}_1 \otimes \tilde{\tilde{R}}_1 \oplus \tilde{\tilde{W}}_2 \otimes \tilde{\tilde{R}}_2 \oplus \tilde{\tilde{W}}_3 \otimes \tilde{\tilde{R}}_3] \oslash [\tilde{\tilde{W}}_1 \oplus \tilde{\tilde{W}}_2 \oplus \tilde{\tilde{W}}_3] \\ &= [(\tilde{\tilde{W}}_1^L \otimes \tilde{\tilde{R}}_1^L, \tilde{\tilde{W}}_1^U \otimes \tilde{\tilde{R}}_1^U) \oplus (\tilde{\tilde{W}}_2^L \otimes \tilde{\tilde{R}}_2^L, \tilde{\tilde{W}}_2^U \otimes \tilde{\tilde{R}}_2^U) \oplus (\tilde{\tilde{W}}_3^L \otimes \tilde{\tilde{R}}_3^L, \tilde{\tilde{W}}_3^U \otimes \tilde{\tilde{R}}_3^U)] \end{aligned}$$

Table 1
A 9-member linguistic term set [3]

Linguistic terms	Generalized fuzzy numbers
Absolutely-low	(0.0, 0.0, 0.0, 0.0; 1.0)
Very-low	(0.0, 0.0, 0.02, 0.07; 1.0)
Low	(0.04, 0.1, 0.18, 0.23; 1.0)
Fairly-low	(0.17, 0.22, 0.36, 0.42; 1.0)
Medium	(0.32, 0.41, 0.58, 0.65; 1.0)
Fairly-high	(0.58, 0.63, 0.80, 0.86; 1.0)
High	(0.72, 0.78, 0.92, 0.97; 1.0)
Very-high	(0.93, 0.98, 1.0, 1.0; 1.0)
Absolutely-high	(1.0, 1.0, 1.0, 1.0; 1.0)

Table 2
Evaluating values of the sub-components $\boldsymbol{A}_1, \boldsymbol{A}_2$, and $\boldsymbol{A}_3$

Sub-components $\boldsymbol{A}_i$	Severity of loss $\tilde{\tilde{W}}_i$	Probability of failure $\tilde{\tilde{R}}_i$
$\boldsymbol{A}_1$	[(0.04, 0.1, 0.18, 0.23; 0.8), (0.04, 0.1, 0.18, 0.23; 1.0)]	[(0.04, 0.1, 0.18, 0.23; 0.7), (0.04, 0.1, 0.18, 0.23; 1.0)]
$\boldsymbol{A}_2$	[(0.58, 0.63, 0.80, 0.86; 0.9), (0.58, 0.63, 0.80, 0.86; 1.0)]	[(0.32, 0.41, 0.58, 0.65; 0.8), (0.32, 0.41, 0.58, 0.65; 1.0)]
$\boldsymbol{A}_3$	[(0.0, 0.0, 0.02, 0.07; 0.7), (0.0, 0.0, 0.02, 0.07; 1.0)]	[(0.72, 0.78, 0.92, 0.97; 0.9), (0.72, 0.78, 0.92, 0.97; 1.0)]

$$
\begin{aligned}
&\oslash [(\tilde{\tilde{W}}_1^L, \tilde{\tilde{W}}_1^U) \oplus (\tilde{\tilde{W}}_2^L, \tilde{\tilde{W}}_2^U) \oplus (\tilde{\tilde{W}}_3^L, \tilde{\tilde{W}}_3^U)] \\
= &[(\tilde{\tilde{W}}_1^L \otimes \tilde{\tilde{R}}_1^L \oplus \tilde{\tilde{W}}_2^L \otimes \tilde{\tilde{R}}_2^L \oplus \tilde{\tilde{W}}_3^L \otimes \tilde{\tilde{R}}_3^L, \tilde{\tilde{W}}_1^U \otimes \tilde{\tilde{R}}_1^U \oplus \tilde{\tilde{W}}_2^U \otimes \tilde{\tilde{R}}_2^U \oplus \tilde{\tilde{W}}_3^U \otimes \tilde{\tilde{R}}_3^U)] \\
&\oslash [(\tilde{\tilde{W}}_1^L \oplus \tilde{\tilde{W}}_2^L \oplus \tilde{\tilde{W}}_3^L, \tilde{\tilde{W}}_1^U \oplus \tilde{\tilde{W}}_2^U \oplus \tilde{\tilde{W}}_3^U)] \\
= &[((0.04, 0.1, 0.18, 0.23; 0.8) \otimes (0.04, 0.1, 0.18, 0.23; 0.7) \oplus (0.58, 0.63, 0.80, 0.86; 0.9) \\
&\otimes (0.32, 0.41, 0.58, 0.65; 0.8) \oplus (0.0, 0.0, 0.02, 0.07; 0.7) \\
&\otimes (0.72, 0.78, 0.92, 0.97; 0.9), (0.04, 0.1, 0.18, 0.23; 1.0) \\
&\otimes (0.04, 0.1, 0.18, 0.23; 1.0) \oplus (0.58, 0.63, 0.80, 0.86; 1.0) \\
&\otimes (0.32, 0.41, 0.58, 0.65; 1.0) \oplus (0.0, 0.0, 0.02, 0.07; 1.0) \\
&\otimes (0.72, 0.78, 0.92, 0.97; 1.0))] \oslash [((0.04, 0.1, 0.18, 0.23; 0.8) \\
&\oplus (0.58, 0.63, 0.80, 0.86; 0.9) \oplus (0.0, 0.0, 0.02, 0.07; 0.7), (0.04, 0.1, 0.18, 0.23; 1.0) \\
&\oplus (0.58, 0.63, 0.80, 0.86; 1.0) \oplus (0.0, 0.0, 0.02, 0.07; 1.0))] \\
= &[(0.1614, 0.2683, 0.7052, 1.095; 0.7), (0.1614, 0.2683, 0.7052, 1.095; 1.0)].
\end{aligned}
$$

[*Step* 2] Based on formula (5), formula (6) and Table 1, we can obtain the COG points of the 9-member linguistic term set, as shown in Table 3. Based on formulas (8)–(11), we can get the COG point $\mathrm{COG}(\tilde{\tilde{R}}^L) = (x^*_{\tilde{\tilde{R}}^L}, y^*_{\tilde{\tilde{R}}^L})$ of the lower fuzzy number $\tilde{\tilde{R}}^L$ and the COG point $\mathrm{COG}(\tilde{\tilde{R}}^U) = (x^*_{\tilde{\tilde{R}}^U}, y^*_{\tilde{\tilde{R}}^U})$ of the upper fuzzy number $\tilde{\tilde{R}}^U$, shown as follows:

$$
\begin{aligned}
y^*_{\tilde{\tilde{R}}^L} &= \frac{\hat{w}^L_{\tilde{\tilde{R}}} \times \left(\frac{a_3^L - a_2^L}{a_4^L - a_1^L} + 2\right)}{6} \\
&= \frac{0.7 \times \left(\frac{0.7052 - 0.2683}{1.095 - 0.1614} + 2\right)}{6} \\
&= 0.2879,
\end{aligned}
$$

Table 3
COG points of the 9-member linguistic term set

Linguistic terms	COG points (x^*, y^*)
Absolutely-low	(0, 0.5)
Very-low	(0.0255, 0.3810)
Low	(0.1370, 0.4035)
Fairly-low	(0.2929, 0.4267)
Medium	(0.4892, 0.4192)
Fairly-high	(0.7178, 0.4345)
High	(0.8471, 0.4267)
Very-high	(0.9745, 0.3810)
Absolutely-high	(1.0, 0.5)

$$
\begin{aligned}
x^*_{\tilde{\tilde{R}}^L} &= \frac{y^*_{\tilde{\tilde{R}}^L}(a_3^L + a_2^L) + (a_4^L + a_1^L)(\hat{w}^L_{\tilde{\tilde{R}}} - y^*_{\tilde{\tilde{R}}^L})}{2\hat{w}^L_{\tilde{\tilde{R}}}} \\
&= \frac{0.2879(0.7052 + 0.2683) + (1.095 + 0.1614)(0.7 - 0.2879)}{2 \times 0.7} \\
&= 0.57, \\
y^*_{\tilde{\tilde{R}}^U} &= \frac{\hat{w}^U_{\tilde{\tilde{R}}} \times \left(\frac{a_3^U - a_2^U}{a_4^U - a_1^U} + 2\right)}{6} \\
&= \frac{1 \times \left(\frac{0.7052 - 0.2683}{1.095 - 0.1614} + 2\right)}{6} \\
&= 0.4113, \\
x^*_{\tilde{\tilde{R}}^U} &= \frac{y^*_{\tilde{\tilde{R}}^U}(a_3^U + a_2^U) + (a_4^U + a_1^U)(\hat{w}^U_{\tilde{\tilde{R}}} - y^*_{\tilde{\tilde{R}}^U})}{2\hat{w}^U_{\tilde{\tilde{R}}}} \\
&= \frac{0.4113(0.7052 + 0.2683) + (1.095 + 0.1614)(1 - 0.4113)}{2 \times 1} \\
&= 0.57.
\end{aligned}
$$

Based on formula (12), we can get the degrees of similarity between the lower fuzzy number $\tilde{\tilde{R}}^L$ and each linguistic term in the 9-member linguistic terms set, shown as follows:

$$
\begin{aligned}
&S(\tilde{\tilde{R}}^L, \text{ absolutely-low}) = 0.2512, \\
&S(\tilde{\tilde{R}}^L, \text{ very-low}) = 0.3478, \\
&S(\tilde{\tilde{R}}^L, \text{ low}) = 0.4092, \\
&S(\tilde{\tilde{R}}^L, \text{ fairly-low}) = 0.4905, \\
&S(\tilde{\tilde{R}}^L, \text{ medium}) = 0.5825, \\
&S(\tilde{\tilde{R}}^L, \text{ fairly-high}) = 0.5199, \\
&S(\tilde{\tilde{R}}^L, \text{ high}) = 0.4617, \\
&S(\tilde{\tilde{R}}^L, \text{ very-high}) = 0.4256, \\
&S(\tilde{\tilde{R}}^L, \text{ absolutely-high}) = 0.3105.
\end{aligned}
$$

In the same way, based on formula (13), we can get the degrees of similarity between the upper fuzzy number $\tilde{\tilde{R}}^U$ and each linguistic term in the 9-member linguistic terms set, shown as follows:

$$S(\tilde{\tilde{R}}^U, \text{absolutely-low}) = 0.3589,$$
$$S(\tilde{\tilde{R}}^U, \text{very-low}) = 0.4262,$$
$$S(\tilde{\tilde{R}}^U, \text{low}) = 0.5626,$$
$$S(\tilde{\tilde{R}}^U, \text{fairly-low}) = 0.7007,$$
$$S(\tilde{\tilde{R}}^U, \text{medium}) = 0.8321,$$
$$S(\tilde{\tilde{R}}^U, \text{fairly-high}) = 0.7428,$$
$$S(\tilde{\tilde{R}}^U, \text{high}) = 0.6595,$$
$$S(\tilde{\tilde{R}}^U, \text{very-high}) = 0.5215,$$
$$S(\tilde{\tilde{R}}^U, \text{absolutely-high}) = 0.4435.$$

Based on formula (14), we can calculate the degree of similarity $S(\tilde{\tilde{R}}^L, \text{absolutely-low})$ between the interval-valued fuzzy number $\tilde{\tilde{R}}$ and the linguistic term "absolutely-low" as follows:

$$\begin{aligned} S(\tilde{\tilde{R}}, \text{absolutely-low}) &= \sqrt{S(\tilde{\tilde{R}}^L, \text{absolutely-low}) \times S(\tilde{\tilde{R}}^U, \text{absolutely-low})} \\ &= \sqrt{0.2512 \times 0.3589} \\ &= 0.3003. \end{aligned}$$

In the same way, based on formula (14), we can calculate the degrees of similarity between the interval-valued fuzzy number $\tilde{\tilde{R}}$ and the other linguistic terms shown in Table 1, shown as follows:

$$S(\tilde{\tilde{R}}, \text{very-low}) = 0.385,$$
$$S(\tilde{\tilde{R}}, \text{low}) = 0.4798,$$
$$S(\tilde{\tilde{R}}, \text{fairly-low}) = 0.5863,$$
$$S(\tilde{\tilde{R}}, \text{medium}) = 0.6962,$$
$$S(\tilde{\tilde{R}}, \text{fairly-high}) = 0.6214,$$
$$S(\tilde{\tilde{R}}, \text{high}) = 0.5518,$$
$$S(\tilde{\tilde{R}}, \text{very-high}) = 0.4711,$$
$$S(\tilde{\tilde{R}}, \text{absolutely-high}) = 0.3711.$$

Because $S(\tilde{\tilde{R}}, \text{medium}) = 0.6962$ has the largest value, the interval-valued fuzzy number $\tilde{\tilde{R}}$ is translated into the linguistic term "medium". That is, the probability of failure of the component $\boldsymbol{A}$ is medium.

5. Conclusions

In this paper, we have presented a new method for handling fuzzy risk analysis problems based on measures of similarity between interval-valued fuzzy numbers. First, we propose a similarity measure to calculate the degree of similarity between interval-valued fuzzy numbers. The proposed similarity measure uses the concept of geometry to calculate the center-of-gravity (COG) points of the lower fuzzy numbers and the upper fuzzy numbers of interval-valued fuzzy numbers, respectively, to calculate the degree of similarity between interval-valued fuzzy numbers. We also prove some properties of the proposed similarity measure. Then, we use the proposed similarity measure of interval-valued fuzzy numbers for handling fuzzy risk analysis problems. The proposed method is more flexible and more intelligent than the methods presented in [1–5] due to the fact that it uses interval-valued fuzzy numbers rather than fuzzy numbers or generalized fuzzy numbers for handling fuzzy risk analysis problems. It provides us with a useful way for handling fuzzy risk analysis problems.

Acknowledgement

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Appendix

Property 1. *Two interval-valued fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ are identical if and only if* $S(\tilde{\tilde{A}}, \tilde{\tilde{B}}) = 1$.

Proof. (i) If $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ are identical, then $a_1^L = b_1^L, a_2^L = b_2^L, a_3^L = b_3^L, a_4^L = b_4^L, a_1^U = b_1^U, a_2^U = b_2^U, a_3^U = b_3^U, a_4^U = b_4^U, \hat{w}_{\tilde{\tilde{A}}}^L = \hat{w}_{\tilde{\tilde{B}}}^L$ and $\hat{w}_{\tilde{\tilde{A}}}^U = \hat{w}_{\tilde{\tilde{B}}}^U$. Based on formula (7), we can see that the degree of similarity $S(\tilde{\tilde{A}}^L, \tilde{\tilde{B}}^L)$ between the lower fuzzy numbers $\tilde{\tilde{A}}^L$ and $\tilde{\tilde{B}}^L$, and the degree of similarity $S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U)$ between the upper fuzzy numbers $\tilde{\tilde{A}}^U$ and $\tilde{\tilde{B}}^U$ are calculated as follows:

$$\begin{aligned} S(\tilde{\tilde{A}}^L, \tilde{\tilde{B}}^L) &= \left[\left(1 - \frac{\sum_{i=1}^{4}\left|a_i^L - b_i^L\right|}{4}\right) \times \left(1 - \left|x^*_{\tilde{\tilde{A}}^L} - x^*_{\tilde{\tilde{B}}^L}\right|\right)\right]^{\frac{1}{2}} \times \frac{\min(y^*_{\tilde{\tilde{A}}^L}, y^*_{\tilde{\tilde{B}}^L})}{\max(y^*_{\tilde{\tilde{A}}^L}, y^*_{\tilde{\tilde{B}}^L})} \\ &= [(1-0)\times(1-0)]^{\frac{1}{2}} \times 1 \\ &= 1, \end{aligned}$$

$$\begin{aligned} S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U) &= \left[\left(1 - \frac{\sum_{i=1}^{4}\left|a_i^U - b_i^U\right|}{4}\right) \times \left(1 - \left|x^*_{\tilde{\tilde{A}}^U} - x^*_{\tilde{\tilde{B}}^U}\right|\right)\right]^{\frac{1}{2}} \times \frac{\min(y^*_{\tilde{\tilde{A}}^U}, y^*_{\tilde{\tilde{B}}^U})}{\max(y^*_{\tilde{\tilde{A}}^U}, y^*_{\tilde{\tilde{B}}^U})} \\ &= [(1-0)\times(1-0)]^{\frac{1}{2}} \times 1 \\ &= 1. \end{aligned}$$

Therefore, we can get

$$\begin{aligned} S(\tilde{\tilde{A}}, \tilde{\tilde{B}}) &= \sqrt{S(\tilde{\tilde{A}}^L, \tilde{\tilde{B}}^L) \times S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U)} \\ &= \sqrt{1 \times 1} \\ &= 1. \end{aligned}$$

(ii) If $S(\tilde{\tilde{A}}, \tilde{\tilde{B}}) = 1$, then $S(\tilde{\tilde{A}}^L, \tilde{\tilde{B}}^L) = 1$ and $S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U) = 1$. Based on formula (7), we can see that if $S(\tilde{\tilde{A}}^L, \tilde{\tilde{B}}^L) = 1$, then

$$\begin{aligned} S(\tilde{\tilde{A}}^L, \tilde{\tilde{B}}^L) &= \left[\left(1 - \frac{\sum_{i=1}^{4}\left|a_i^L - b_i^L\right|}{4}\right) \times \left(1 - \left|x^*_{\tilde{\tilde{A}}^L} - x^*_{\tilde{\tilde{B}}^L}\right|\right)\right]^{\frac{1}{2}} \times \frac{\min(y^*_{\tilde{\tilde{A}}^L}, y^*_{\tilde{\tilde{B}}^L})}{\max(y^*_{\tilde{\tilde{A}}^L}, y^*_{\tilde{\tilde{B}}^L})} \\ &= 1. \end{aligned}$$

It implies that $a_1^L = b_1^L, a_2^L = b_2^L, a_3^L = b_3^L, a_4^L = b_4^L, x^*_{\tilde{\tilde{A}}^L} = x^*_{\tilde{\tilde{B}}^L}, y^*_{\tilde{\tilde{A}}^L} = y^*_{\tilde{\tilde{B}}^L}$ and $\hat{w}^L_{\tilde{\tilde{A}}} = \hat{w}^L_{\tilde{\tilde{B}}}$. Therefore, the lower fuzzy numbers $\tilde{\tilde{A}}^L$ and $\tilde{\tilde{B}}^L$ are identical. In the same way, if $S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U) = 1$, then

$$S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U) = \left[\left(1 - \frac{\sum_{i=1}^{4} \left|a_i^U - b_i^U\right|}{4}\right) \times \left(1 - \left|x^*_{\tilde{\tilde{A}}^U} - x^*_{\tilde{\tilde{B}}^U}\right|\right)\right]^{\frac{1}{2}} \times \frac{\min(y^*_{\tilde{\tilde{A}}^U}, y^*_{\tilde{\tilde{B}}^U})}{\max(y^*_{\tilde{\tilde{A}}^U}, y^*_{\tilde{\tilde{B}}^U})}$$
$$= 1.$$

It implies that $a_1^U = b_1^U, a_2^U = b_2^U, a_3^U = b_3^U, a_4^U = b_4^U, x^*_{\tilde{\tilde{A}}^U} = x^*_{\tilde{\tilde{B}}^U}, y^*_{\tilde{\tilde{A}}^U} = y^*_{\tilde{\tilde{B}}^U}$ and $\hat{w}^U_{\tilde{\tilde{A}}} = \hat{w}^U_{\tilde{\tilde{B}}}$. Therefore, the upper fuzzy numbers $\tilde{\tilde{A}}^U$ and $\tilde{\tilde{B}}^U$ are identical. □

Property 2. $S(\tilde{\tilde{A}}, \tilde{\tilde{B}}) = S(\tilde{\tilde{B}}, \tilde{\tilde{A}})$.

Proof. Based on formula (14), we can see that

$$S(\tilde{\tilde{A}}, \tilde{\tilde{B}}) = \sqrt{S(\tilde{\tilde{A}}^L, \tilde{\tilde{B}}^L) \times S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U)},$$
$$S(\tilde{\tilde{B}}, \tilde{\tilde{A}}) = \sqrt{S(\tilde{\tilde{B}}^L, \tilde{\tilde{A}}^L) \times S(\tilde{\tilde{B}}^U, \tilde{\tilde{A}}^U)},$$

where

$$S(\tilde{\tilde{A}}^L, \tilde{\tilde{B}}^L) = \left[\left(1 - \frac{\sum_{i=1}^{4} \left|a_i^L - b_i^L\right|}{4}\right) \times \left(1 - \left|x^*_{\tilde{\tilde{A}}^L} - x^*_{\tilde{\tilde{B}}^L}\right|\right)\right]^{\frac{1}{2}} \times \frac{\min(y^*_{\tilde{\tilde{A}}^L}, y^*_{\tilde{\tilde{B}}^L})}{\max(y^*_{\tilde{\tilde{A}}^L}, y^*_{\tilde{\tilde{B}}^L})},$$

$$S(\tilde{\tilde{B}}^L, \tilde{\tilde{A}}^L) = \left[\left(1 - \frac{\sum_{i=1}^{4} \left|b_i^L - a_i^L\right|}{4}\right) \times \left(1 - \left|x^*_{\tilde{\tilde{B}}^L} - x^*_{\tilde{\tilde{A}}^L}\right|\right)\right]^{\frac{1}{2}} \times \frac{\min(y^*_{\tilde{\tilde{B}}^L}, y^*_{\tilde{\tilde{A}}^L})}{\max(y^*_{\tilde{\tilde{B}}^L}, y^*_{\tilde{\tilde{A}}^L})},$$

$$S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U) = \left[\left(1 - \frac{\sum_{i=1}^{4} \left|a_i^U - b_i^U\right|}{4}\right) \times \left(1 - \left|x^*_{\tilde{\tilde{A}}^U} - x^*_{\tilde{\tilde{B}}^U}\right|\right)\right]^{\frac{1}{2}} \times \frac{\min(y^*_{\tilde{\tilde{A}}^U}, y^*_{\tilde{\tilde{B}}^U})}{\max(y^*_{\tilde{\tilde{A}}^U}, y^*_{\tilde{\tilde{B}}^U})},$$

$$S(\tilde{\tilde{B}}^U, \tilde{\tilde{A}}^U) = \left[\left(1 - \frac{\sum_{i=1}^{4} \left|b_i^U - a_i^U\right|}{4}\right) \times \left(1 - \left|x^*_{\tilde{\tilde{B}}^U} - x^*_{\tilde{\tilde{A}}^U}\right|\right)\right]^{\frac{1}{2}} \times \frac{\min(y^*_{\tilde{\tilde{B}}^U}, y^*_{\tilde{\tilde{A}}^U})}{\max(y^*_{\tilde{\tilde{B}}^U}, y^*_{\tilde{\tilde{A}}^U})},$$

where $\sum_{i=1}^4 |a_i^L - b_i^L| = \sum_{i=1}^4 |b_i^L - a_i^L|$, $\sum_{i=1}^4 |a_i^U - b_i^U| = \sum_{i=1}^4 |b_i^U - a_i^U|$, $|x^*_{\tilde{\tilde{A}}^L} - x^*_{\tilde{\tilde{B}}^L}| = |x^*_{\tilde{\tilde{B}}^L} - x^*_{\tilde{\tilde{A}}^L}|$, $|x^*_{\tilde{\tilde{A}}^U} - x^*_{\tilde{\tilde{B}}^U}| = |x^*_{\tilde{\tilde{B}}^U} - x^*_{\tilde{\tilde{A}}^U}|$, $\frac{\min(y^*_{\tilde{\tilde{A}}^L}, y^*_{\tilde{\tilde{B}}^L})}{\max(y^*_{\tilde{\tilde{A}}^L}, y^*_{\tilde{\tilde{B}}^L})} = \frac{\min(y^*_{\tilde{\tilde{B}}^L}, y^*_{\tilde{\tilde{A}}^L})}{\max(y^*_{\tilde{\tilde{B}}^L}, y^*_{\tilde{\tilde{A}}^L})}$ and $\frac{\min(y^*_{\tilde{\tilde{A}}^U}, y^*_{\tilde{\tilde{B}}^U})}{\max(y^*_{\tilde{\tilde{A}}^U}, y^*_{\tilde{\tilde{B}}^U})} = \frac{\min(y^*_{\tilde{\tilde{B}}^U}, y^*_{\tilde{\tilde{A}}^U})}{\max(y^*_{\tilde{\tilde{B}}^U}, y^*_{\tilde{\tilde{A}}^U})}$. Thus, we can see that $S(\tilde{\tilde{A}}^L, \tilde{\tilde{B}}^L) = S(\tilde{\tilde{B}}^L, \tilde{\tilde{A}}^L)$ and $S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U) = S(\tilde{\tilde{B}}^U, \tilde{\tilde{A}}^U)$. Therefore, we can see that $S(\tilde{\tilde{A}}, \tilde{\tilde{B}}) = S(\tilde{\tilde{B}}, \tilde{\tilde{A}})$. □

Property 3. *If $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ are two real numbers, then $S(\tilde{\tilde{A}}, \tilde{\tilde{B}}) = 1 - |a - b|$.*

Proof. If $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ are two real numbers, then we can see that

$$\begin{aligned}
\tilde{\tilde{A}} &= [(a_1^L, a_2^L, a_3^L, a_4^L; \hat{w}^L_{\tilde{\tilde{A}}}), (a_1^U, a_2^U, a_3^U, a_4^U; \hat{w}^U_{\tilde{\tilde{A}}})] \\
&= [(a, a, a, a; 1), (a, a, a, a; 1)] \\
&= (a, a, a, a; 1) \\
&= a, \\
\tilde{\tilde{B}} &= [(b_1^L, b_2^L, b_3^L, b_4^L; \hat{w}^L_{\tilde{\tilde{B}}}), (b_1^U, b_2^U, b_3^U, b_4^U; \hat{w}^U_{\tilde{\tilde{B}}})] \\
&= [(b, b, b, b; 1), (b, b, b, b; 1)] \\
&= (b, b, b, b; 1) \\
&= b.
\end{aligned}$$

Based on formulas (8) and (10), we can see that $y^*_{\tilde{\tilde{A}}^L} = y^*_{\tilde{\tilde{B}}^L} = y^*_{\tilde{\tilde{A}}^U} = y^*_{\tilde{\tilde{B}}^U} = 1/2$. Based on formulas (12) and (13), we can get

$$\begin{aligned}
S(\tilde{\tilde{A}}^L, \tilde{\tilde{B}}^L) &= \left[\left(1 - \frac{\sum_{i=1}^{4} \left|a_i^L - b_i^L\right|}{4}\right) \times \left(1 - \left|x^*_{\tilde{\tilde{A}}^L} - x^*_{\tilde{\tilde{B}}^L}\right|\right)\right]^{\frac{1}{2}} \times \frac{\min(y^*_{\tilde{\tilde{A}}^L}, y^*_{\tilde{\tilde{B}}^L})}{\max(y^*_{\tilde{\tilde{A}}^L}, y^*_{\tilde{\tilde{B}}^L})} \\
&= \left[\left(1 - \frac{4\,|a - b|}{4}\right) \times (1 - |a - b|)\right]^{\frac{1}{2}} \times 1 \\
&= 1 - |a - b|, \\
S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U) &= \left[\left(1 - \frac{\sum_{i=1}^{4} \left|a_i^U - b_i^U\right|}{4}\right) \times \left(1 - \left|x^*_{\tilde{\tilde{A}}^U} - x^*_{\tilde{\tilde{B}}^U}\right|\right)\right]^{\frac{1}{2}} \times \frac{\min(y^*_{\tilde{\tilde{A}}^U}, y^*_{\tilde{\tilde{B}}^U})}{\max(y^*_{\tilde{\tilde{A}}^U}, y^*_{\tilde{\tilde{B}}^U})} \\
&= \left[\left(1 - \frac{4\,|a - b|}{4}\right) \times (1 - |a - b|)\right]^{\frac{1}{2}} \times 1 \\
&= 1 - |a - b|.
\end{aligned}$$

Therefore, we can see that

$$\begin{aligned}
S(\tilde{\tilde{A}}, \tilde{\tilde{B}}) &= \sqrt{S(\tilde{\tilde{A}}^L, \tilde{\tilde{B}}^L) \times S(\tilde{\tilde{A}}^U, \tilde{\tilde{B}}^U)} \\
&= \sqrt{(1 - |a - b|) \times (1 - |a - b|)} \\
&= 1 - |a - b|. \quad \square
\end{aligned}$$

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